

November 30, 2012

On the dynamical entropy of dense QCD states

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We discuss dense states of QCD matter formed in high-energy hadronic and heavy-ion collisions from the point of view of statistical physics of non-equilibrium processes. For this sake, we first propose a formulation of the *dynamical entropy* of dense QCD states in the “saturation regime” leading to a color glass condensate (CGC). The statistical physics description amounts to describe the modification of the color correlation length with energy as a *compression* process for which nonequilibrium thermodynamic properties are applicable. We derive an expression of the dynamical entropy in terms of the rapidity evolution of the unintegrated gluon distributions in the colliding nuclei, verifying suitable positivity and irreversibility properties. We extend this approach to the initial pre-equilibrium (glasma) state of an heavy-ion collision. It allows for a definition of the initial entropy before the evolution towards the hydrodynamic regime as a function of the glasma correlation length and an overlap parameter characterizing the low-momentum spectrum of the glasma state. This initial entropy, by extension to the $\mathcal{N} = 4$ SYM theory, is then matched as the key input parameter to the strong coupling evaluation of thermalization towards the hydrodynamic regime based on the AdS/CFT correspondence. It thus allows to cast a bridge between the weak and strong coupling phases of an heavy-ion reaction.

PACS numbers:

I. INTRODUCTION

It is commonly believed that statistical physics concepts may be relevant to describe the development and outcome of ultra-high energy collisions between incident hadrons or nuclei. For instance, the multi-production of particles during such a high-energy reaction has been related to the entropy produced by the collision. A statistical physics framework seems especially relevant for the central distribution of multiplicities in ultra-relativistic heavy-ion collisions, where relativistic hydrodynamics, assumed during a transient evolution of the quark-gluon plasma (QGP), give a nice account [1] of the observable features of the reaction.

However, the theoretical difficulty remains in relating these features to the basics of Quantum Chromodynamics (QCD), being the field-theoretical framework where, ultimately at least, those statistical physics features, and in particular the entropy release, could be properly defined. Indeed, one faces both difficult statistical-physics and field-theoretic problems, since the initial reaction medium is both far from equilibrium and submitted to a time-dependent QCD evolution in a strong-coupling regime.

Recent interesting steps forward made in this direction are the following. In the context of strong gauge coupling studies using the AdS/CFT correspondence [2], it was shown [3] that starting from various initial conditions, the thermalization leading to a hydrodynamic behavior of a strongly-coupled gauge theory plasma essentially depends on an “initial entropy” factor defined within the gauge/gravity duality framework using involved notions of black hole (BH) thermodynamics. The hydrodynamic regime is identified with the long-time behavior of a receding BH [4]. In this case, the entropy release during the evolution towards the hydrodynamic regime may be associated with the difference between the final hydrodynamic entropy and the abovementioned “initial entropy”. This encouraging result has yet some limitations. It applies to a maximally supersymmetric theory in the strong coupling regime, while one is in practice dealing with QCD with weak-coupling initial states in the context of the Color Glass Condensate [5] and the formation of an initial *glasma* [6]. Hence the problem is open to make the matching between a weak coupling initial phase and the strong coupling scenario of thermalization. Our goal is to know how to define properly the “initial entropy” for dense QCD states already at weak coupling.

In this context, a recent paper [7] introduces the notion of a thermodynamical entropy associated with the production

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of gluons in the saturation regime of dense CGC initial states in proton-proton collisions¹. It is motivated by the relation between the saturation scale (denoted by Q_s in transverse momentum) with a temperature of thermally produced hadrons and a dynamically generated mass, itself also related to the saturation scale Q_s . This, together with the expected relation between entropy and the inclusive gluon production in a dense-dilute hadron-hadron collision, leads to an estimate of the entropy of a dense CGC initial state in the framework of the Golec-Biernat Wüsthoff model [9] for the QCD unintegrated gluon distribution (UGD).

Ref.[7] is a stimulating approach, since it proposes a *macroscopic* approach to the entropy of a CGC ensemble of gluons, in a thermodynamic context. It may be hindered by the assumptions of thermodynamic equilibrium, such as the existence of a temperature and an entropy which are not yet well defined in principle in a perturbative QCD framework which seems to govern far-from-equilibrium processes where these notions are not standard. However, the notion of *dense* QCD states, which is basic in the CGC approach, has specific features which may lead to a convenient statistical physics description of the high-energy collision processes without making assumptions on thermodynamic equilibrium. In some sense it would be useful to have a *microscopic* definition of entropy, analogous to the Boltzmann statistical mechanic approach, based on the QCD dynamics of the CGC. We will propose such a notion of a *dynamical entropy* for dense QCD states in the present paper.

In the next section II, we will give the definition of a “dynamical entropy” for dense QCD states. Then in section III, we will apply it to models of the CGC medium, and check the matching with the macroscopic approach of Ref.[7]. In the next section IV, we will propose an extension of the dynamical entropy approach to the initial *glasma* phase of heavy-ion collisions and analyze its matching with the thermalization scenario based on the AdS/CFT correspondence. A final section V is devoted to conclusions with both a summary of results and outlook.

II. DYNAMICAL ENTROPY OF CGC STATES

Dense QCD states may be present in various physical situations. In this section, we will consider the CGC medium. It plays a role in Deep Inelastic Scattering (DIS) in the “small- x ” range where it represents the quantum states of the target as seen by the virtual photon when the Bjorken variable $x \sim Q^2/e^Y \ll 1$, Q^2 being the photon virtuality and Y is the total rapidity range of the final states. It also appears in the description of “dense-dilute” collisions such as proton-nucleus at very high energies. Later on, we will extend our analysis to the initial medium in ultra-relativistic heavy-ion reactions obtained by the collision of two boosted nuclei in their center-of-mass frame.

These dense QCD states may be theoretically described within QCD at weak coupling, through the nonlinear energy evolution starting from a dilute partonic state. The QCD evolution with total rapidity Y increases the density of partons and finally reaches the saturation regime leading to a CGC state. It is characterized among other features [5], by a limiting transverse size $R_s(Y) \sim 1/Q_s(Y)$, where $Q_s(Y)$ is the rapidity-dependent saturation momentum. $R_s(Y)$ is in fact the color correlation length at given rapidity.

As energy increases, the parton density becomes high enough such that individual parton branching is compensated by their recombination. This leads to many interesting properties which can be evaluated still in the (resumed at leading or next-leading logarithms) weak coupling regime of QCD. Among them “geometric scaling” states [10] that the unintegrated gluon distribution $\phi(k; Y)$ of a CGC medium essentially depends only on one variable implying the saturation scale. It verifies a scaling property as a function of a single variable $u = k^2 R_s^2(Y)$, within some controlled approximation [11, 12], namely

$$\phi(k, Y) d^2k \sim \phi(u = k^2 R_s^2) R_s^2 d^2k. \quad (1)$$

The balance between parton branching and recombination at saturation and geometric scaling will be the main property allowing for a statistical mechanic approach, despite the probable absence of thermal equilibrium. Let us consider this QCD compensation mechanism between branching and recombination from the statistical physics point-of-view on [13–20] (choosing some useful references inside an abundant recent statistical physics literature, see *e.g.* [20]).

The CGC distribution of gluons can be obtained by the evolution of a partonic state (essentially gluons) from a given rapidity Y_1 to a dense medium at higher rapidity Y_2 , characterized by a spectrum in transverse momentum k defined by the UGD $\phi(k, Y_2)$. Using the formulation of classical² statistical physics, the partons forming the CGC medium may be interpreted as a *stationary* state, since, at each value of rapidity gluon branching and recombination

¹ A different approach to entropy production in heavy-ion collisions can be found in Refs.[8].

² Though obtained through a quantum field theory framework, the system of partons generated by the QCD evolution has been proven to be described at leading logarithm approximation as a classical branching and recombination process.

compensate each other in the saturation regime. The evolution in rapidity $Y_1 \rightarrow Y_2$ inside the dense regime can be considered as an (*a priori* nonequilibrium) process driven by a “dynamical parameter”, being the energy increase $\exp Y_1/2 \rightarrow \exp Y_2/2$ of the partonic system.

Indeed, in the statistical physics framework, a similar situation is met when a distribution of stationary states is defined through a probability distribution $\mathcal{P}^{\text{Stat}}(z; \lambda)$, where z is a general notation for the phase space, see [16, 20] (and for a brief reminder, Appendix B). By changing the dynamical parameter $\lambda_1 \rightarrow \lambda_2$, the system evolves in time leading to a final distribution which can be compared to a new stationary distribution $\mathcal{P}^{\text{Stat}}(z; \lambda_2)$. A key property of this nonequilibrium process is the Hatano-Sasa identity involving both $\mathcal{P}^{\text{Stat}}(z; \lambda_{1,2})$, see Eq.(B1).

Indeed, we are to show that the branching-recombining mechanism, responsible for the *geometric-scaling* property of the unintegrated gluon distribution $\phi(k, R_s) \sim \phi(k R_s(Y))$ gives rise to a transverse-momentum probability distribution verifying the analogous of the Hatano-Sasa identity (B1).

We are led to the following identification (see Appendix B):

$$z \rightsquigarrow k, \quad \lambda \rightsquigarrow Y, \quad \mathcal{P}^{\text{Stat}}(z; \lambda) dz \rightsquigarrow \mathcal{P}(k, Y) d^2k, \quad (2)$$

where, using (1), the transverse momentum probability distribution of gluons in the CGC is defined as

$$\mathcal{P}(k, Y) d^2k = \frac{\phi(k, Y) d^2k}{\int \phi(k, Y) d^2k} \Rightarrow \mathcal{P}(u) du, \quad (3)$$

the last relation coming from the geometric scaling form (1). This definition ensures the normalization condition

$$\int \mathcal{P}(k; Y) d^2k = \int \mathcal{P}(u) du \equiv 1. \quad (4)$$

It is now straightforward to write the identity

$$\begin{aligned} \left\langle \exp - \left[\int_{Y_1}^{Y_2} \frac{d}{dY} \{ \log \mathcal{P}(k, Y) \} dY \right] \right\rangle_{Y_2} &= \int \mathcal{P}(k, Y_2) d^2k \times \exp - \left[\log \left\{ \frac{\mathcal{P}(k, Y_2)}{\mathcal{P}(k, Y_1)} \right\} \right] \\ &\equiv \int \mathcal{P}(k, Y_1) d^2k \equiv 1, \end{aligned} \quad (5)$$

where the left-hand side correlator $\langle \dots \rangle_{Y_2}$ is defined by averaging over the probability distribution in the final state at Y_2 and making use of the probability normalization relation (4) at Y_1 .

As we shall demonstrate further on, Eq. (5) appears as the QCD version of the Hatano-Sasa identity (B1) for a CGC ensemble of states. It is a general property of QCD saturation and the CGC [5] that the gluons organize themselves in cells of typical saturation scale size. This is made quite explicitly, when introducing the geometric scaling formulation of (1). Hence, the rapidity increase is responsible of the shrinkage of the color correlation length which appears as a *compression* process understood as the mean-field result of the denser gluonic medium. It gives rise to a modification of the probability distribution of gluon momentum which verifies the relation (5). This is explicit from (1), where the compression is manifest in the variation of the saturation scale $R_s(Y_1) \rightarrow R_s(Y_2) < R_s(Y_1)$.

Thanks to this interpretation, and following the proposed identification (2), we introduce the the notion of *dynamical entropy* as follows. We define the *dynamical entropy* density of a CGC parton medium at rapidity Y_2 , coming from the QCD evolution in rapidity $Y_1 \rightarrow Y_2$ inside a transverse area of order of the initial color correlation size $R_s^2(Y_1)$ by

$$\Sigma^{Y_1 \rightarrow Y_2} = \left\langle \log \frac{\mathcal{P}(k, Y_2)}{\mathcal{P}(k, Y_1)} \right\rangle_{Y_2} \equiv \int \mathcal{P}(k, Y_2) d^2k \log \left\{ \frac{\mathcal{P}(k, Y_2)}{\mathcal{P}(k, Y_1)} \right\}. \quad (6)$$

We note that in the statistical physics literature formulas similar to (6) appear also (see *e.g.* [19]) as the *relative entropy* of a probability distribution w.r.t. a reference one (here $\mathcal{P}(k, Y_2)$ compared to $\mathcal{P}(k, Y_1)$). It amounts to quantify the “amount of disorder” provoked by the rapidity evolution $Y_1 \rightarrow Y_2$ of the CGC medium. This notion is well adapted to the CGC medium of gluons, for which the competition between branching and recombination is expected to increase with the densification of gluons.

Let us list the role and properties of the dynamical entropy formula (6).

i) *Positivity*. By construction, the identity (5), using the well-known [21] Jensen inequality $e^{\langle X \rangle} \leq \langle e^X \rangle$, for any probabilistic distribution here identified to $X \equiv -\log \frac{\mathcal{P}(k, Y_2)}{\mathcal{P}(k, Y_1)}$, leads to the positivity condition

$$\Sigma^{Y_1 \rightarrow Y_2} = \left\langle \log \frac{\mathcal{P}(k, Y_2)}{\mathcal{P}(k, Y_1)} \right\rangle_{Y_2} \geq -\log \left\langle \exp \left\{ -\log \left[\frac{\mathcal{P}(k, Y_2)}{\mathcal{P}(k, Y_1)} \right] \right\} \right\rangle_{Y_2} = 0, \quad (7)$$

for all $Y_2 \geq Y_1$. Hence, the familiar positivity condition is obtained for any increase in total rapidity.

ii) *Geometric scaling case in terms of non-equilibrium thermodynamics*. Considering now the geometric scaling property (1), the identity (5) may also be discussed in thermodynamic terms, following the discussion and properties of Jarzynski identity, see Appendix A and (A1). Indeed, using the geometric scaling version of (1), the identity (5) can be rewritten in geometric-scaling variables where for short notation $R_{1,2} \equiv R_1(Y_{1,2})$, one gets

$$\left\langle \exp - \left\{ \log \frac{\mathcal{P}(R_2^2 k^2)}{\mathcal{P}(R_1^2 k^2)} \right\} \right\rangle_{R_2} = \exp \left\{ -\log \frac{R_2^2}{R_1^2} \right\} \equiv \frac{Q_s^2(Y_2)}{Q_s^2(Y_1)}, \quad (8)$$

where we reintroduced the saturation momentum scales $Q_s(Y_{1,2}) = 1/R_s^2(Y_{1,2})$.

The relation (8) can be put in tight analogy with the Jarzynski identity [13] of the thermodynamics of far-from-equilibrium processes (see Appendix A) which writes

$$\left\langle e^{-\mathcal{W}/T} \right\rangle_B = e^{-(F_C - F_A)/T}, \quad (9)$$

where the average is made over different realizations of a process driving rapidly an equilibrium state A at temperature T to an out-of-equilibrium state B , which thermalizes towards a new equilibrium state C (keeping the driving dynamical parameter constant at its new value) at the same temperature.

The Jarzynski identity relates the stochastic distribution of thermodynamical works in the process $A \rightarrow B$ to the free energy balance ΔF between the two equilibrium states $A \rightarrow C$. Interestingly, the amount of dissipative work $\mathcal{W}^{Diff} \equiv \mathcal{W} - \Delta F$ during the process $A \rightarrow B$ is then related to the entropy production $\Delta S = (\langle \mathcal{W} \rangle - \Delta F)/T \geq 0$, if the state B is able to relax towards the temperature T , keeping the driving parameter constant.

Comparing now (8) to (9), one realizes that the expression $\log R_2^2/R_1^2$ corresponds to the logarithm of a ratio of the available phase-space for dimension R_2^2 vs. R_1^2 . It is the free energy change of the particle of an ideal gas contained inside a 2-dimensional “box” when the size changes by compression $R_1 \rightarrow R_2 < R_1$. Hence, a thermodynamic interpretation of the QCD relation (5) is that the modification of the total rapidity $Y_1 \rightarrow Y_2$ induces a modification of the CGC ensemble of states with reduced saturation size R_2 (or equivalently increased mean momentum Q_2), resulting in an entropy eventually generated by further on relaxation. One is led to the analogy

$$\begin{aligned} \log \frac{\mathcal{P}(R_2^2 k^2)}{\mathcal{P}(R_1^2 k^2)} &\rightsquigarrow \frac{\mathcal{W}}{T} \\ \log \frac{R_1^2}{R_2^2} &\rightsquigarrow \frac{\Delta F}{T} \\ \log \frac{\mathcal{P}(k, Y_2)}{\mathcal{P}(k, Y_1)} &\rightsquigarrow \frac{\mathcal{W} - \Delta F}{T} \equiv \frac{\mathcal{W}^{Diff}}{T} \\ \Sigma^{R_1 \rightarrow R_2} = \left\langle \log \frac{\mathcal{P}(k, Y_1)}{\mathcal{P}(k, Y_2)} \right\rangle_{Y_2} &\rightsquigarrow \Delta S, \end{aligned} \quad (10)$$

where ΔS is, in the thermodynamic context, the *entropy production* (by gluonic degree of freedom) due to the compression $R_1 \rightarrow R_2$ when the system relaxes to a state at the same temperature T as the initial one but within the restricted domain size R_2 .

The last line of (10) is suggestive of a special relation between the *dynamical entropy* of a CGC state Σ and the entropy production after relaxation ΔS . We shall confirm this analogy for Gaussian gluon distributions, by an explicit calculation in the next section.

iii) *Dynamical entropy of a physical CGC state*. The dynamical entropy formula (6) remains written for a unit gluon degree of freedom within the color correlation area and for an arbitrary rapidity evolution $Y_1 \rightarrow Y_2$ within

the QCD saturation domain. In order to go to CGC states one deals with in high-energy collisions between hadrons and/or nuclei, one is due to extend the formalism to those physical cases.

Using the thermodynamic analogy, the right-hand side of the identity (8) can be interpreted as coming from the standard free energy of one gluon d.o.f. inside of an ideal gas of gluons confined in a radial transverse box of saturation size R_s , as expressed in the second row of (10). In order to determine the overall dynamical entropy that we may associate to a physical CGC initial state (*e.g.* hadronic or nuclear one), we will assume that the probability distribution of reference $\mathcal{P}(k, Y_1)$ considers an initial “transverse cell” at rapidity Y_1 where R_1 is a typical color correlation length for hadronic matter (proton or nucleus) at rest. In fact, this reference distribution will not play a role for large enough rapidity.

Then, we consider the boosted CGC medium when $Y_1 \rightarrow Y_2$, $R_1 \rightarrow R_2$, where Y_2 is the rapidity at which one observes (*e.g.* by deep-inelastic or dense-dilute hadronic scattering) the corresponding CGC state. In this framework, by addition of the individual contributions (6), the total dynamical entropy density dS_T/dy for a boosted target CGC state will be the sum over all degrees of freedom, namely color multiplicity $N_c^2 - 1$, standard [22] gluon occupation number $\sim 1/4\pi N_c \alpha_s$ in longitudinal coordinate space. For the transverse degrees of freedom, one has to take into account the average number R_T^2/R_2^2 of “transverse cells” at initial rapidity Y_1 , where R_T is the hadronic or nucleus target size, and finally the average number μ of gluonic degrees of freedom inside a cell which will be determined later on using the Gaussian models for the UGD’s.

$$\frac{dS}{dy} = \frac{N_c^2 - 1}{4\pi N_c \alpha_s} \cdot \frac{R_T^2}{R_1^2} \cdot \mu \cdot \Sigma^{R_1 \rightarrow R_2} . \quad (11)$$

Gluon correlation effects beyond the ideal gas approximation may induce correction factors, which we do not discuss in the present study.

III. APPLICATION TO GAUSSIAN CGC MODELS

We want now to apply our concept of dynamical entropy *cf.* (6) to simple CGC models in order to describe its main properties in a concrete way. Another motivation is to confront our definition of dynamical entropy for a physical CGC state Eq.(11) to the one proposed in Ref.[7], which comes from the computation of the amount of gluon production, starting from a dense-dilute configuration of initial protons.

Let us apply our approach to Gaussian models of unintegrated gluon distributions, in which we introduce a parameter generalization of the GBW model [9] in order to clarify the contributions to the entropy. The GBW model indeed is used by [7] for its definition of entropy which we wish to compare with our formulation. We thus consider unintegrated gluon distributions $\phi(k, Y)$ (1) with a Gaussian tail, namely

$$\begin{aligned} \mathcal{P}(k, Y) d^2k &\Rightarrow \mathcal{P}(u = R_s^2 k^2) du = \psi(u) e^{-u} du \\ &\sim \Gamma^{-1}(\kappa) u^{\kappa-1} e^{-u} du , \end{aligned} \quad (12)$$

where the non-Gaussian prefactor $\psi(u)$, describing the behavior in the low- k range of gluon momentum, is more simply parametrized by a power-like term $u^{\kappa-1}$. We shall call κ the “overlap parameter” for reasons becoming clear later on in this section. Note that $\kappa = 2$, which is the GBW value, corresponds to a dipole distribution verifying the transparency property at zero dipole size [9].

Definition (12) is indeed a slight generalization of the unintegrated gluon distribution coming from the Golec-Biernat Wusthoff dipole model [7, 9] obtained for the value $\kappa = 2$.

Starting by implementing (12) in the dynamical entropy definition (6), one gets

$$\Sigma^{Y_1 \rightarrow Y_2} \equiv \left\langle \log \frac{\mathcal{P}(R_2^2 k^2)}{\mathcal{P}(R_1^2 k^2)} - \log \frac{R_1^2}{R_2^2} \right\rangle_{Y_2} = \langle k^2 \rangle_{Y_2} (R_1^2 - R_2^2) - \left\langle \log \frac{R_1^2 \cdot \psi(R_1^2 k^2)}{R_2^2 \cdot \psi(R_2^2 k^2)} \right\rangle_{Y_2} \geq 0 , \quad (13)$$

and, more explicitly for the κ -dependent family (12)

$$\begin{aligned} \Sigma^{Y_1 \rightarrow Y_2} &= \int \left[k^2 (R_1^2 - R_2^2) - \kappa \log \frac{R_1^2}{R_2^2} \right] \mathcal{P}(k^2 R_2^2) R_2^2 dk^2 \\ &= \langle k^2 \rangle_{Y_2} (R_1^2 - R_2^2) - \kappa \log (R_1^2/R_2^2) = \kappa \{ (R_1^2/R_2^2 - 1) - \log (R_1^2/R_2^2) \} \geq 0 . \end{aligned} \quad (14)$$

One can easily check the strict final inequality (14) for all $Y_2 > Y_1$, *i.e.* $R_2 < R_1$ while it goes smoothly to zero (as $\kappa/2 (R_1^2/R_2^2 - 1)^2$) when $R_2 \rightarrow R_1$.

Let us list relevant properties of formula (14).

- *Dynamical entropy of the CGC state.* Using the relation (11), we can compute the total entropy density $dS^{Y_1 \rightarrow Y_2} / \pi R_T^2 dy$ for a CGC state of overall transverse size R_T . Assuming the framework of an initial distribution of independent transverse “cells” with size R_1 , it is given by

$$\frac{1}{\pi R_T^2} \frac{dS^{Y_1 \rightarrow Y_2}}{dy} = \frac{N_c^2 - 1}{4\pi^2 N_c \alpha_s} \frac{1}{R_1^2} \cdot \mu \cdot \Sigma^{Y_1 \rightarrow Y_2} = \kappa \mu \frac{N_c^2 - 1}{4\pi^2 N_c \alpha_s} \left\{ Q_2^2 - Q_1^2 \left(1 + \log \frac{Q_2^2}{Q_1^2} \right) \right\}, \quad (15)$$

expressed in terms of the saturation momentum dependence $Q_s = 1/R_s$. Note that at large Q_2^2/Q_1^2 , formula (15) shows that the target dynamical entropy by unit of transverse area and rapidity becomes independent of the initial condition at Y_1 and is directly proportional to Q_s^2 . The logarithmic corrections provide a smooth transition to quasi reversibility $dS \rightarrow 0$ when $Q_2^2/Q_1^2 \rightarrow 1$.

- *“Microscopic” vs. “macroscopic” entropy: determination of the μ parameter.* Let us now derive the dynamical CGC entropy density precisely for the case considered in Ref.[7], and check the matching with the expression obtained in [7] from gluon production in the dense-dilute collision case. We tend to call it “macroscopic” entropy since it relies on a derivation from thermodynamic concepts. In our case, that we call “microscopic” we proceed from a counting of the configurations in transverse momentum phase-space parametrized by the probability $\mathcal{P}(k, Y)$, from (12).

For p - p scattering, considered in [7], we use, for simplicity, (15) for the GBW gluon distribution with $\kappa=2$. We find for the dynamical (“microscopic”) entropy

$$\frac{1}{A_T} \frac{dS}{dy} = \frac{2\mu(N_c^2 - 1)}{\pi^2 \alpha_s N_c} Q_2^2 \left(1 + \mathcal{O} \left[\frac{Q_1^2}{Q_2^2} \log \frac{Q_2^2}{Q_1^2} \right] \right), \quad (16)$$

with $\kappa = 2$ and the proton mean transverse area $A_T \equiv \pi R_T^2/4$.

This results correctly matches the one³ obtained in the derivation of ref. [7] which also gives the same behavior as the first term in formula (16), independent of the initial condition. Hence the two derivations of the entropy of a CGC medium are consistent with each other despite their marked difference. Indeed, one comes from the gluon production process in a dilute-dense collision and the other is based on the configurations of the CGC medium itself.

Moreover, for an exact matching between the two formulas, one is led to identify

$$\mu \equiv \frac{3\pi}{2}, \quad (18)$$

which gives a non trivial estimate of the number of effectively independent gluonic degrees of freedom within a given color correlation length R_s . Interestingly, it comes as a pure number, which may be characteristic of the average number of gluons inside a colorless cluster of a CGC medium generated by saturation. It would be worth checking that this remains valid for more elaborated UGD’s.

- *Strong irreversibility of the formation of a CGC state.* It is interesting to note that, using the same method, one can compute from (6) the dynamical entropy $\Sigma^{R_2 \rightarrow R_1}$ associated with the backward process, *i.e* the expansion $R_2 \rightarrow R_1 > R_2$. By exchanging $R_2 \leftrightarrow R_1$ in (14) one finds

$$\Sigma^{R_2 \rightarrow R_1} \equiv \left\langle \log \frac{\mathcal{P}(k, Y_1)}{\mathcal{P}(k, Y_2)} \right\rangle_{Y_1} = \kappa \{ \log (R_1^2/R_2^2) - (1 - R_2^2/R_1^2) \} \geq 0. \quad (19)$$

It is easy to check also in that case the strict inequality for $R_1 > R_2$, with a same smooth behavior towards the equality as $R_2 \rightarrow R_1$. However, one finds a logarithmic behavior $\log (R_1^2/R_2^2)$ by contrast with the quadratic power-like behavior(14). In fact, in that case the main contribution to the dissipative work is due to the free energy term in (10), while the “work” contribution is almost zero. The interpretation is that (*cf.* [18]) only few gluons within the stochastic distribution of transverse momenta feel the expansion of the color correlation

³ The relevant formula (25) in [7] reads, in our notations

$$\frac{1}{A_T} \frac{dS}{dy} = \frac{3(N_c^2 - 1)}{\pi \alpha_s N_c} Q_s^2 + cst., \quad (17)$$

length. Hence the “gluon compression” mechanism is in general much more dissipative and thus irreversible than “gluon expansion”. This is even more remarkable, when assuming a “cyclic” process, namely a compression $R_1 \rightarrow R_2$ followed by the reverse expansion $R_2 \rightarrow R_1$. The overall balance of entropy $\Delta\Sigma^\odot(R_1, R_2)$, following Eqs.(14,19) writes

$$\Delta\Sigma^\odot(R_1, R_2) = \kappa \frac{(R_1^2 - R_2^2)^2}{R_1^2 R_2^2} \sim \kappa \frac{R_1^2}{R_2^2}, \quad (20)$$

where the last approximation is for $R_1 \gg R_2$ large. Formula (20) makes explicit the irreversible feature of the color correlation size changing processes⁴.

From this, one can induce that the QCD saturation mechanism due to the high density of gluons is generating a large entropy and thus highly dissipative in thermodynamic language. Hence the dynamical entropy (6) seems to capture interesting irreversible physical features of the formation of dense QCD states, and eventually, through the cyclic process, their relaxation towards equilibrium.

- *Comparison with a 1-dimensional ideal gas model moved by a piston.* It is quite instructive to compare the results for the QCD two-dimensional dynamical entropy (6) with the non-relativistic one-dimensional ideal gas model of Ref.[18]. One considers the rapid action $L_i \rightleftharpoons L_f < L_i$ of a piston on a longitudinal box containing an ideal gas of independent particles, starting with temperature T . One can compute the work distribution due to the piston either by expansion [18] or compression [24]. The piston model is a well-known far-from-equilibrium example where the Jarzynski identity (9) can be explicitly verified, and one finds for the average dissipative work $\langle \mathcal{W} \rangle - \Delta F$ (and thus the dynamical entropy)

$$\begin{aligned} \Sigma^{L_1 \rightarrow L_2} &\equiv \frac{1}{T} \{ \langle \mathcal{W} \rangle - \Delta F^{L_1 \rightarrow L_2} \} \propto L_1/L_2 - 1 - \log(L_1/L_2) \\ \Sigma^{L_1 \leftarrow L_2} &\equiv \frac{1}{T} \{ \langle \mathcal{W} \rangle - \Delta F^{L_2 \rightarrow L_1} \} \sim -\Delta F^{L_2 \rightarrow L_1} \propto \log(L_1/L_2). \end{aligned} \quad (21)$$

It is tempting to interpret our results (14) and (19) as the radial analogue of the two equations of (21). In some sense, the QCD field-theoretical saturation mechanism can be thought of as due to multiple shocks of individual gluons on the effective “walls” due to the shrinkage of the color correlation length, as a mean-field effect on an individual gluon reproducing the many-body interactions of gluons at saturation. Rapid expansion is much less entropy producing, since it creates almost no work, leaving only, as a dissipative mechanism, the logarithmic free-energy cost $-\Delta F$. As a side remark, it is interesting to figure out that saturated gluons within a small color correlation “cell” may behave as non relativistic particles, if they acquire an effective mass $M \sim Q_s$ as suggested in [7].

- *Role of the “overlap factor” κ .* The factor κ , which parameterizes the low- k range of the gluon transverse momentum distribution (*cf.* (12)), appears as an overall multiplicative factor in our resulting formulas for the dynamical entropy (14,15,19) without modifying the quadratic dependence in Q_s^2 . It is linked to the structure of the gluon probability distributions $\mathcal{P}(k, Y_2)$ *vs.* $\mathcal{P}(k, Y_1)$, *cf.* (14). Indeed, a smaller (*resp.* larger) value of κ corresponds to wider (*resp.* less) overlap between the two distributions and thus lower (*resp.* higher) amount of dynamical entropy. It appears to be much dependent on the low- p_T range of the UGD’s.

A remark here is adequate concerning the Weizsäcker-Williams gluon distribution, which is also discussed in the framework of UGD’s [25]. The Weizsäcker-Williams gluon distribution has the physical interpretation as the number density of gluons inside the hadron/nucleus in light-cone gauge, but is not appearing in the cross-section observables [25]. The main difference with the one used in (15) (and also in [7]) is the infra-red logarithmic behavior for $k^2 \rightarrow 0$, would correspond to an overlap factor $\kappa \approx 1$ in (14). In our definition (15) of the dynamical entropy, we choose the “dipole gluon distribution” and formula (15) for the following reasons: it corresponds to a suppression of the color radiation at null transverse momentum, and a neat maximum at the saturation momentum Q_s , which is in a good analogy with the classical notion of a gas of particles in a “box” of size $R_s = 1/Q_s$, which makes consistent the aspects of classical far-from equilibrium statistical physics on which relies our definition. This is related to the fact that QCD dipoles are colorless states which follow *classical* dynamics, at least at leading logs order of the perturbative expansion.

⁴ Such “cyclic” process could be eventually describing the entropy production obtained through the formation and subsequent relaxation of a CGC state, when evaluated at weak QCD coupling. It would be interesting to relate it to the entropy production through thermalization.

IV. EXTENSION TO THE DENSE INITIAL STATE OF HEAVY-ION REACTIONS.

The pre-equilibrium state of matter reached immediately after the heavy-ion collision, is only a transient one. In the most common scenario, it emanates from the interaction of the two CGC states representing the wave-function of the incident colliding particles, forming “at time zero” what is thought to be in the literature the *glasma* phase [6]. Then it evolves towards the hydrodynamic QGP phase, during a strikingly short-time period, as indicated by simulations, which is referred to as plasma *thermalization*. This transition is still the less understood part of the QCD plasma formation in heavy-ion reactions. We shall investigate how the notion of a dynamical entropy may provide some new light on the questions raised by the rapid thermalization found in heavy-ion collisions.

The two incident nuclei considered in the center-of-mass frame are boosted and thus can be described as two independent CGC states. Hence, our formalism may in principle be applied to the initial CGC states of both nuclei. However, one has to take into account the *glasma* state formed from these CGC states at the interaction time. Hence, the application of our method defined in the previous section requires some extension. In particular, the significant matching with the “macroscopic entropy” of Ref.[7], see previous section, was made considering “dense-dilute” collision while the *glasma* state corresponds to a “dense-dense” configuration in the overall center-of-mass frame. In the following, we shall make a proposal how one can extend the notion of dynamical entropy for the glasma and its impact on the thermalization problem.

- *Dynamical entropy of the glasma.*

Let us consider the glasma phase of Ref. [28], as an application example of our approach. Extending the dynamical entropy concept to the glasma defined in [28] appears nontrivial, since one has to consider a classical Yang-Mills field calculation taking as initial value the result of the JIMWLK evolution for each nucleus. Those calculations lead to a prediction for the gluon distribution in the glasma $\phi_{glasma}(p_\perp, y)$ where p_\perp is the transverse gluon momentum and y , the rapidity evolution away from the initial condition at $y = 0$, taken in [28] as the McLerran-Venugopalan model [23]. What is numerically observed is that the resulting glasma spectrum satisfies geometric scaling using the gluon (*i.e.* QCD adjoint) saturation momentum $Q_s(Y)$. We are thus led to apply a similar recipe as in the CGC case, with the difference that new nonlinear effects due to the classical Yang-Mills calculations are present, especially in the low p_\perp range. One thus may proceed as follows.

Defining a probability distribution, similarly to (3),

$$\mathcal{P}_{glasma}(p_\perp, Y) d^2p_\perp \equiv \frac{\phi_{glasma}(p_\perp, Y) d^2k}{\int \phi_{glasma}(p_\perp, Y) d^2p_\perp} \Rightarrow \mathcal{P}_{glasma}(u = p_\perp^2/Q_s^2) du, \quad (22)$$

one is led to propose for the glasma dynamical entropy density (in a transverse cell of order $R \sim 1/Q_s(Y)$, for the evolution $Y_1 \rightarrow Y_2$

$$\Sigma_{glasma}^{Y_1 \rightarrow Y_2} \equiv \left\langle \log \frac{\mathcal{P}_{glasma}(p_\perp, Y_2)}{\mathcal{P}_{glasma}(p_\perp, Y_1)} \right\rangle_{Y_2} \geq 0. \quad (23)$$

Assuming for simplicity a Gaussian form of the distribution, we consider a rapidity evolution of the glasma state in the range where geometric scaling with saturation momentum $Q_s(Y)$ is valid. Taking also into account in a phenomenological way the observed depletion [28] of gluon radiation in the small range of p_\perp , one may write a formula similar to (12) being used for the CGC states in section III, namely

$$\mathcal{P}^{glasma} d^2p_\perp = \psi(p_\perp/Q_s) e^{-p_\perp^2/Q_s^2} d^2p_\perp \approx \frac{1}{\Gamma(\kappa_{gl})} \left(\frac{p_\perp^2}{Q_s^2} \right)^{\kappa_{gl}-1} e^{-p_\perp^2/Q_s^2} d^2p_\perp, \quad (24)$$

where the last approximation comes from considering the same input as in Ref.[7], *i.e.* (12) and the small- p_\perp regularization prefactor $\psi(p_\perp/Q_s)$ being approximated using an appropriate overlap factor κ_{gl} . Applying now our definition (6) we get

$$\Sigma_{glasma}^{Y_1 \rightarrow Y_2} \sim \kappa_{gl} Q_2^2/Q_1^2, \quad (25)$$

where, as usual $Q_i \equiv Q_s(Y_i)$. It would be interesting to make a more precise evaluation of the glasma dynamical entropy and compare it with the sum of entropies of the initial CGC states for the same rapidity increase. This requires to go beyond the Gaussian toy model we are using and we postpone this to further study.

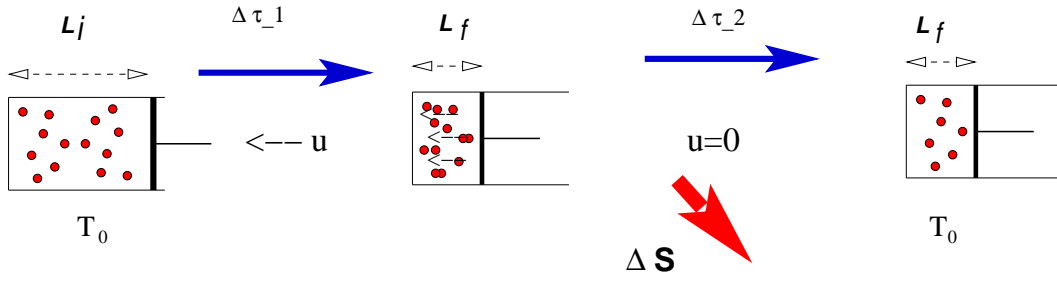


FIG. 1: *Piston model: Compression and Thermalization.* Left: Thermal state at temperature T_0 and length L_i ; Middle: Far-from equilibrium compressed state. Right: Thermalization through entropy production to the heat bath and relaxation to a denser thermal state at temperature T and length L_f .

- *Entropy production and thermalization in heavy-ion collisions*

The pre-equilibrium state of matter reached immediately after the heavy-ion collision, is only a transient one. It evolves, during a quite short time as indicated by simulations, towards the hydrodynamic QGP phase which will give rise finally to hadronization. This stage, driving the interacting medium from the CGC initial nucleus states through the glasma formation to the hydrodynamic behavior of the QGP, called *thermalization* is not well understood, due to lack of a *microscopic* theory of thermalization based on QCD. It may require a strong coupling treatment of collisions in QCD which is not yet theoretically available. Some hints have been obtained [3] from the AdS/CFT correspondence for the similar problem formulated in the case of the maximally symmetric $\mathcal{N}=4$ gauge field theory.

In this context, let us examine what can be obtained from the formulation of the dynamical entropy for the initial state before thermalization, using the previous results. For this sake, we will confront both *macroscopic* and *microscopic* points of view.

From a *macroscopic* point-of-view inspired by out-of-equilibrium statistical physics, there exists an intimate relation between the dynamical entropy of an initial state as defined in our approach and the entropy production during thermalization. This has already been discussed in the case of the dilute-dense configuration of the initial states, see section III. Our goal now is to extend it to the dense-dense case which is typical for heavy-ion collisions in the central rapidity region. Second we will consider a *microscopic* approach, namely the strong coupling scenario of Ref.[3], where the notion of an “initial entropy” has been introduced using a dual gravitational version of the $\mathcal{N}=4$ gauge field theory. We will conjecture that its determination for QCD may be based on the dynamical entropy of the glasma state.

Coming back for this sake to the Jarzynski and Hatano-Sasa identities. A key feature is their interpretation in terms of the entropy generated by a fast non-equilibrium process in a finite time, and released to the environment by relaxation. In the case of the Jarzynski identity (9), see Appendix B, and assuming a thermodynamic coupling with an outside heat bath at temperature T , the exponent is related to the amount of average dissipative work $\langle \mathcal{W}^{diff} \rangle$, itself connected to the production of entropy ΔS by the relation

$$\Delta S \equiv \frac{1}{T} \langle \mathcal{W}^{diff} \rangle \equiv \frac{1}{T} (\langle \mathcal{W} \rangle - \Delta F) \geq 0. \quad (26)$$

In practice, the produced entropy comes from the relaxation of the out-of-equilibrium state by freezing the external force acting on the system, which finally towards thermal equilibrium at the initial value of the temperature T .

This process is, among others, well exemplified by the exact results obtained for one-dimensional piston model [18], see Fig.1. Starting from the thermal state of an ideal gas at temperature T in a box of initial length L_i , the action of the piston, with speed u during a short time $\Delta\tau_1$, shrinks the box size till a final size $L_i \rightarrow L_f = L - |u|\Delta\tau_1$ which is then kept fixed, leaving the system relax towards the initial temperature⁵. Indeed, the action of the piston gives rise to the far-from-equilibrium state of Fig.1 middle, with the particles of gas being far from thermal equilibrium since most part of them acquires momentum and kinetic energy by shocks on the piston [18]. Then its thermalization through the production of entropy makes the system relaxing to the thermal state at temperature T and length L_f , depicted in Fig.1 right.

⁵ Another interesting exact model is the *adiabatic* expansion and compression of a dilute interacting gas [17]

It is interesting to note that the estimate (25) of the dynamical entropy of the initial glasma state appears as a simple extension of relation (26) with

$$\Delta S = \frac{1}{T} \langle \mathcal{W}^{diff} \rangle \rightsquigarrow \Sigma^{Y_1 \rightarrow Y_2} \geq 0 ; \quad \frac{1}{T} \langle \mathcal{W} \rangle \rightsquigarrow \kappa_{gl} \left(\frac{Q_2^2}{Q_1^2} - 1 \right) ; \quad \frac{1}{T} \Delta F \rightsquigarrow \kappa_{gl} \log \left(\frac{Q_2^2}{Q_1^2} \right) , \quad (27)$$

for a two-dimensional version of the piston model. The physical interpretation is that the evolution of the glasma state is obtained from the *compression* of gluons confined within a box of color correlation length $R_1 \rightarrow R_2$ due to the increasing rate of branching and recombination of gluons at higher density.

The precise determination of ΔS asks for a mechanism incorporating the expansion of the QCD matter, the thermalization process, the subsequent hydrodynamic evolution and eventually the hadronization of the QGP. While such a theoretical treatment is still unknown for QCD (beyond phenomenological approaches), a refined treatment of thermalization through the hydrodynamic evolution could be obtained in the strong coupling scheme of Ref.[3] for the $\mathcal{N}=4$ SYM theory. Let us then consider this case in the context of our approach. It has been shown that the boost-invariant proper-time expansion of the $\mathcal{N}=4$ SYM plasma leads to a hydrodynamic stage [4] with small viscosity over entropy ratio. Numerical solutions of the early-time evolution using the gravity dual allows one to examine the properties of the pre-equilibrium evolution, starting with various initial conditions which are not specified in [3], and thus not related to the glasma states. We try and fill this gap in the sequel.

The main common feature of the solutions of the strong coupling studies using the gauge/gravity duality, beyond the variety of initial conditions, are the short proper-time evolution towards the hydrodynamic stage, before which they differ. Those regularities, such as entropy variation and the thermalization temperature seem to only depend on the initial energy and entropy densities, namely

$$\epsilon(0) = N_c^2 \cdot \frac{3}{8} \pi^2 \cdot T_{eff}^4(0) \quad ; \quad \frac{1}{A_T} \frac{dS}{d\eta} = N_c^2 \cdot \frac{1}{2} \pi^2 \cdot T_{eff}^2(0) \cdot s(0) \quad (28)$$

where one introduced an “initial effective temperature” $T_{eff}(0)$ and an “initial entropy” density $s(0)$ by degree of freedom, both defined by (28) from the corresponding energy and entropy densities by unit of space-time rapidity η and transverse area A_T .

Note that the initial state *is not* at equilibrium and thus $T_{eff}(0)$ *is not* a physical temperature and $s(0)$ *is not* the usual entropy at equilibrium. Following a notion of “dynamical entropy” in general relativity, it is defined *via* the holographic correspondence as the element of the apparent horizon’s area situated at geodesic distance at proper-time $\tau = 0$ from the origin in the bulk space (see [3] and references therein).

Factorizing the color degrees of freedom, we note that (28) gives rise to the following relation

$$\cdot \frac{1}{A_T} \frac{(3/4) dS/d\eta}{\epsilon(0)} = s(0) \cdot T_{eff}^{-2}(0) , \quad (29)$$

valid for each degree of freedom of the initial state. Note that the correction factor 3/4 is coming from the known ratio (see *e.g.* [2], section 3.6) between the strong coupling entropy at finite temperature and the ideal gas one. We shall not take into account this factor in the following, which deserves more study going beyond our context.

Considering now the corresponding densities for dense QCD initial states, and for application, the Gaussian model (22), we may write formulas in parallel with (28) and (29). For this sake, one starts from the formulas for the glasma energy density⁶ and for the dynamical entropy (25), with the standard counting of gluon degrees of freedom (assuming for sake of simplicity the same occupation number $1/(4\pi^2\alpha_s N_c)$ in rapidity as for CGC states, cancelled in the ratio). One obtains

$$\epsilon_{gl} = \frac{N_c^2 - 1}{4\pi^2\alpha_s N_c} \cdot Q_{gl}^4 \quad ; \quad \frac{1}{A_T} \frac{dS_{gl}}{dy} = \kappa_{gl} \cdot \mu_{gl} \cdot \frac{N_c^2 - 1}{4\pi^2\alpha_s N_c} \cdot Q_{gl}^2 , \quad (30)$$

where ϵ_{gl} is the initial⁷ energy density of the glasma, $1/Q_{gl} = R_{gl}$ is the size of the glasma color correlation length, κ_{gl} the overlap factor in the Gaussian case (24) and μ_{gl} is the number of gluonic degrees of freedom in the color

⁶ For simplicity we use for the energy density in rapidity the standard factor. For a more refined determination, see [27].

⁷ There is a subtlety in the determination of ϵ_{gl} if using as an input the Weizsäcker-Williams definition of the UGD [27] which diverges at $\tau = 0$. We here still use for consistency of the Gaussian approach the “dipole” UGD.

correlation “cell” of the glasma (*cf.* (18) for its determination for the CGC). One finally note that (30) gives rise to the following normalization independent relation

$$\frac{1}{A_T} \frac{dS_{gl}/dy}{\epsilon_{gl}(0)} = \kappa_{gl} \cdot \mu_{gl} \cdot Q_{gl}^{-2} . \quad (31)$$

Matching (31) with (29) can be performed, since the proportionality of $T_{eff}(0)/Q_{gl}$ is expected from the scale invariance of both the $\mathcal{N}=4$ SYM theory (from conformal invariance) and the dense state distributions in QCD at saturation (from geometric scaling).

Now, in order to get an estimate of the initial entropy $s(0)$ taking as an example the CGC case [7], one uses $\kappa_{gl} = 2$, with $T_{eff}(0)/Q_{gl} = 1/(2\pi)$, from the determination of the temperature [29] through the Unruh property. In the Bjorken boost-invariant regime ([1], first reference) one can identify the space-like and energy-momentum definitions of rapidity $\eta \equiv y$. Finally we assume the same value $\mu_{gl} = 3\pi/2$ as in (18) for the number of gluons by color cell. We then find

$$s(0) \rightsquigarrow \kappa_{gl} \cdot \mu_{gl} \cdot (T_{eff}(0)/Q_{gl})^2 \Rightarrow \frac{3}{4\pi} . \quad (32)$$

where the last number comes from the Gaussian form of a CGC state, taken as an example of the dense QCD state. Note that this number is in the ballpark of the range $0 < s(0) < 1/2$ obtained in [3]. It corresponds to a quite sizable entropy density.

More generally, the key order parameter $s(0)$ of the thermalization analysis in $\mathcal{N}=4$ SYM theory is in direct relation with the “overlap parameter” κ_{gl} , characterizing the strength of the dynamical entropy of the glasma state (in $\mathcal{N}=4$ SYM). This is consistent with the entropy increasing with decreasing overlap of probability distributions.

Recalling the main results of the numerical analyses showing [3] that the system evolves quite rapidly towards a hydrodynamic behavior, even before isotropization occurs, the rate of entropy production $s_f - s(0)$ and thermalization proper-time (indeed, better to tell “hydrodynamization time”) τ_{th} in units of $T_{eff}^{-1}(0)$ are driven by the value of $s(0)$. In our corresponding QCD evaluation (32) for the Gaussian model, it is the value of the overlap parameter κ which is relevant. Higher is the dynamic entropy parameter κ , *i.e.* smaller is the overlap between the gluon probability distributions at Q_2 *vs.* Q_1 , where $R_1 = 1/Q_1$ is taken for convenience of order of the standard color correlation length at rest. Then, greater is the initial entropy $s(0)$ and the subsequent increase of entropy $s_f - s(0)$ during thermalization. Following [3], the thermalization time is consequently shorter. Hence, in our framework, it means that the thermalization time depends on the “overlap parameter”, and thus on the strength of the dynamical entropy of the initial dense saturation state.

V. CONCLUSIONS

Let us give our conclusions. We have proposed a definition of a dynamical entropy for dense QCD states of matter at high energies. This definition is based on statistical physics tools which have been shown to be valid for large classes of non-equilibrium processes. Our approach is inspired by the Jarzynski and Hatano-Sasa fluctuation theorems [13, 16] and is applied to the formation of dense QCD matter through an energy-dependent evolution scheme defined in the QCD field theoretical framework at weak coupling. It describes the evolution of the color correlation length $R_1 \rightarrow R_2$, corresponding to the rapidity evolution $Y_1 \rightarrow Y_2$ of a dense QCD medium of gluons which is the result of gluon branching and recombination.

The derivation of an identity (5) provides a guide for the definition of a dynamical entropy functional verifying positivity, similar to the 2^{nd} principle of irreversible thermodynamics. It acquires an interesting physical picture when the gluon distribution of the dense QCD medium reaches the *geometric scaling* regime. In that stage, the boost-invariant distribution of gluons can be described as transverse *cells* whose radial size is constrained by the collective effect of the others to be the *saturation scale* $R_s = 1/Q_s(Y)$. The dynamical entropy is expressed as an overlap functional between the gluon distributions at different total rapidities Y_1 and $Y_2 > Y_1$ and thus saturation sizes $R_s(Y_1)$ and $R_s(Y_2) < R_s(Y_1)$.

Summary

Summarizing concretely our results:

- *Dynamical Entropy of CGC states :*

It is possible to define an entropy functional (by unit gluonic degree of freedom)

$$\Sigma^{Y_1 \rightarrow Y_2} = \left\langle \log \frac{\mathcal{P}(k, Y)}{\mathcal{P}(k, Y_1)} \right\rangle_{Y_2} \equiv \int d^2k \log \left\{ \frac{\mathcal{P}(k, Y_2)}{\mathcal{P}(k, Y_1)} \right\} \mathcal{P}(k, Y_2) \geq 0, \quad (33)$$

where $\mathcal{P}(k, Y)$ is the probability distribution of gluon transverse momentum at rapidity Y . It is defined using the QCD dipole unintegrated gluon distribution after normalization to unity. $\Sigma^{Y_1 \rightarrow Y_2}$ is the dynamical entropy density (by transverse space “cell”) during a far-from-equilibrium “compression” process during which the correlation length R_1 shrinks to $R_2 < R_1$.

In the geometric scaling set-up, the color correlation length of transverse size R_s appears explicitly as

$$\Sigma^{Y_1 \rightarrow Y_2} = \left\langle \log \frac{\mathcal{P}(R_2^2 k^2)}{\mathcal{P}(R_1^2 k^2)} \right\rangle_{Y_2} - \log \frac{R_1^2}{R_2^2} \rightsquigarrow \frac{1}{T} \left\{ \langle \mathcal{W} \rangle^{R_1 \rightarrow R_2} - \Delta F^{R_1 \rightarrow R_2} \right\}. \quad (34)$$

Hence, (33) in the geometric scaling form (34) admits the interpretation of a genuine entropy in a thermodynamical framework, see Appendix A, since the *dissipative work* performed during this process corresponds to the difference between the average work $\langle \log \mathcal{P}(R_2^2 k^2) / \mathcal{P}(R_1^2 k^2) \rangle_{Y_2}$ and the variation of the free energy $\log R_1^2 / R_2^2$ between the initial and final “cells”, during the “compression” process $R_1 \rightarrow R_2$ due to the increase of the gluon density when the total rapidity evolves $Y_1 \rightarrow Y_2$.

In a non-thermodynamic framework, see Appendix B, where the CGC states are assimilated to non-equilibrium “stationary states”, then $\Sigma^{R_1 \rightarrow R_2}$ plays the role of a dynamical entropy acquired by the CGC “initial stationary state” from a rapidity evolution and released through relaxation.

- *Application : the Gaussian model.*

As an example, we considered dipole gluon densities of the form $\phi(k, R_1) \sim (k R_1)^{2(\kappa-1)} \exp(-k^2 R_1^2)$ generalizing the GBW model [9], for which one explicitly finds

$$\Sigma^{R_1 \rightarrow R_2} = \kappa \{ (Q_2^2 / Q_1^2 - 1) - \log (Q_2^2 / Q_1^2) \} \geq 0, \quad (35)$$

where $Q_s(Y_1, Y_2) \equiv 1/R_{(1,2)}$, *i.e.* the saturation transverse momenta at the initial and final stage of the rapidity evolution. The parameter κ is an *overlap factor* characterizing the amount of overlap between the initial and final distributions, especially at low transverse momentum. Greater is κ , smaller is the overlap, and thus stronger is the dynamical entropy density generated by the evolution $Y_1 \rightarrow Y_2$, due to the increasing number of branching and recombination QCD vertices involved in the process.

Interestingly, the result (35) was proved to be identical for large ΔY , up to a factor $\mu \equiv 3\pi/2$, to the evaluation of the entropy variation of a CGC state coming from a *macroscopic* formalism, *i.e.* a thermodynamic interpretation [7] of the distribution of gluons produced in a dense-dilute collision. The characteristic factor μ can be interpreted as the (average) number of gluonic degrees of freedom inside a transverse CGC “cell”. To our knowledge, this is a new result coming from the comparison of two different but complementary approaches to the notion of entropy for a CGC medium. This appears to be in conformity with the statistical physics interpretation, *via* the comparison between *microscopic vs. macroscopic* approaches, which relates the work fluctuations -here given the momentum distribution in the CGC state- to the entropy production - here, the gluon multiplicity produced in a “dense-dilute” collision probing the structure of the dense medium.

- *Dynamical entropy of the glasma in heavy-ion collisions.*

We considered the class of models using the Color Glass Condensate formalism describing the initial nuclei at the collision time and leading to the *glasma* as the initial interacting state of heavy-ion collisions. Using the rapidity evolution of the glasma distribution of gluons, as *e.g.* performed in Refs.[27, 28], it is possible, at least approximatively, to define the dynamical entropy for an rapidity variation δy in a way similar to the CGC states.

As an illustration, we used a phenomenological Gaussian model for the probability distribution, namely

$$\mathcal{P}^{glasma} d^2 p_\perp = \frac{1}{\Gamma(\kappa_{gl})} \left(\frac{p_\perp^2}{Q_s^2} \right)^{\kappa_{gl}-1} e^{-p_\perp^2 / Q_s^2} d^2 p_\perp, \quad (36)$$

where, from the analysis of [28], one verifies geometric scaling with momentum scale Q_s^2 corresponding to the adjoint QCD representation, and a regularization at smaller momentum due to the nonlinear effect of the classical Yang-Mills

fields initiated by the initial CGC states. It is described here by a phenomenological value κ_{gl} of the overlap factor. One then gets in this example at leading order, for one gluonic degree of freedom,

$$\Sigma_{glasma}^{Y_1 \rightarrow Y_2} \sim \kappa_{gl} \left(\frac{Q_s(Y_2)}{Q_s(Y_1)} \right)^2. \quad (37)$$

- *Hints for thermalization:*

The evaluation through QCD calculations of the dynamical entropy for the initial state of heavy-ion collisions allows for interesting hints on the problem of the transition towards the thermalized hydrodynamic stage of the Quark-Gluon Plasma.

Noting that the expression (37) for the dynamical entropy density of a glasma state can be a candidate for the “initial entropy” density $s(0)$ appearing as a key parameter in the strong coupling thermalization mechanism [3] of the ADS/CFT correspondence, one finds the relation

$$s(0) \rightsquigarrow \kappa_{gl} \cdot \mu_{gl} \cdot \left\{ \frac{T_{eff}(0)}{Q_{gl}} \right\}^2, \quad (38)$$

where $T_{eff}(0)/Q_{gl}$ is the ratio between the effective temperature (determined by the initial energy density) and the saturation scale of the glasma, expected to be a universal constant, independent of the total rapidity Y . The parameter μ_{gl} is the number of gluonic degrees of freedom in a glasma “cell”. Assuming for simplicity the same values for κ, μ as for the Gaussian CGC distribution, we find values of $s(0)$ in the range compatible with the AdS/CFT framework of Ref.[3].

It is interesting to see that (38) gives a realization of the main physical idea behind the Jarzynski and Hatano-Sasa identities, namely the connection between the work (and thus momentum) probability distribution and the overall produced entropy by a far-from equilibrium process. Indeed, a large (*resp.* small) overlap factor κ_{gl} means that the gluon momentum variations induced by the rapidity (and thus energy) evolution of a glasma state have been large (*resp.* small). Hence the contribution to the subsequent entropy (and thus particle) production is larger (*resp.* smaller).

In practice, and if the input parameter values are taken from the CGC example in QCD and the AdS/CFT example from in $\mathcal{N} = 4$ SYM theory, we find a rather large initial entropy density, in the range considered in [3]. It remains to see whether more realistic applications, both for QCD and AdS/CFT confirm this first approximation.

Outlook

Many pending questions about non-equilibrium processes occurring during a high-energy heavy-ion collisions which may be discussed using the tools of non-equilibrium statistical mechanics, as we started to perform in the present study. Let us mention some of the prospects in that direction.

- *Dynamical Entropy of dense QCD states : theory*

Considering our application to QCD far-from-equilibrium processes, the property of the stochastic identities (*e.g.* [13, 15, 16]) is the possibility of finding an expression for the entropy -usually related with the *final* multiplicity- using the knowledge of the momentum distribution in the *initial* state. Hence, on the theoretical level, the question arises how the properties of the dynamical entropy are related to the basics of the QCD evolution equations in the saturation setting, namely the BK or JIMWLK equations, see *e.g.* [5]. Indeed, they are known to lead to geometric scaling gluon distributions when the rapidity Y increases, which is used in our derivation. One more specific question is the role of the geometric scaling *violations*, which are known to occur in the full solution of the QCD equations [11]. Hence, one may ask what is the mathematical meaning of the general expression of the dynamical entropy

$$\Sigma \equiv \left\langle \log \frac{\mathcal{P}(k, Y_2)}{\mathcal{P}(k, Y_1)} \right\rangle_{Y_2} \geq 0, \quad (39)$$

where $\mathcal{P}(k; Y_2)$ is proportional (with a suitable normalization) to the derivative of the gluon structure function $F(k, Y)$, solution of the BK/JIMWLK QCD evolution equation as a function of the rapidity Y . This can be addressed as a question about integro-differential equations of F-KPP type which admit asymptotic traveling wave solutions, and are the proper mathematical formulation of geometric scaling [11].

- *Dynamical Entropy of dense QCD states : phenomenology*

On the phenomenological basis, it would be interesting to make a more quantitative estimate of the various entropy contributions during all expected successive steps of a heavy-ion reaction using the phenomenological knowledge one

has acquired about these states. Namely, we can usefully consider physical parameterizations of CGC states and of the glasma phase, and of their evolution towards the hydrodynamic phase. These problems deserve future studies.

Finally, we may hope that the gap of knowledge yet existing for a proper general and operative formulation of nonequilibrium processes in quantum field theories may be reduced thanks to studies on heavy-ion collisions. We conjecture that the existence of generalized work identities may give some hints for elaborating such a general framework, may be in relation with the formalism of Wilson lines and loops, which is rooted in the bases of QCD, both at weak and strong coupling.

• *The thermalization problem*

The formalism and calculation presented in section IV for the initial state entropy of an heavy-ion collision are in fact the *weak coupling* analogous of the strong coupling ones in the framework of the AdS/CFT correspondence performed in Ref.[3]. A comparison between both approaches, see *e.g.* (38), shows that a matching can be obtained between the glasma initial state⁸ and the subsequent evolution within the strong coupling scheme. the question now arises to eventually make a more direct comparison between the definition of entropy in those dynamical regimes: the one coming from the Shannon-type definition (33) and the one coming from the dual gravitational theory related to the local area of the apparent horizon [3]. Our formalism may also be useful in the thermalization approach based on instabilities, *e.g.* Ref.[30].

Appendix A: The Jarzynski identity

Generic identities have been derived when the evolution is fast and thus very far from equilibrium. These are the Jarzynski [13] and Hatano-Sasa [16] (also Crooks [15]) fluctuation relations, which we now briefly describe in the text.

It is well-known from statistical physics textbooks that inequalities involving the average work $\langle \mathcal{W} \rangle$ performed by a statistical system verifies thermodynamic *inequalities* related to the second principle of thermodynamics, namely the increase of the total entropy for an isolated system. For instance, for the thermodynamic transition between two equilibrium states, one automatically verifies the celebrated second principle of thermodynamics

$$\Delta S = \frac{1}{T}(\langle \mathcal{W} \rangle - \Delta F) \geq 0 \equiv \frac{1}{T} \times \langle \mathcal{W}_{Diff} \rangle \geq 0, \quad (\text{A1})$$

where ΔS is the increase of entropy, T , the temperature, and \mathcal{W} is the work performed in the medium *i.e.* $\Delta E \equiv \langle \mathcal{W} \rangle$ the total energy change, ΔF the variation of the free energy of the system. In such a relation, $\mathcal{W} - \Delta F = \mathcal{W}_{Diss}$ is the *dissipative work* paid by the system during a nonequilibrium process. The equality to zero in (A1) is only for reversible processes, through slow dissipation-less evolution. However, as in particular shown in by the work identities [13] there are necessarily fluctuations with $\mathcal{W}_{Diss} < 0$ in the dissipative work distribution [13].

To be a bit more precise in this short introduction, the Jarzynski identity relates the stochastic distribution of thermodynamical works in the process $A \rightarrow B$ to the free energy balance ΔF between the two equilibrium states $A \rightarrow C$. Interestingly, the amount of dissipative work $\mathcal{W}^{Diff} \equiv \mathcal{W} - \Delta F$ during the process $A \rightarrow B$ is then related to the entropy production $\Delta S = 1/T(\langle \mathcal{W} \rangle - \Delta F) \geq 0$, if the state B is able to relax towards the temperature T , keeping the driving parameter constant.

The Jarzynski identity [13] reads

$$\left\langle e^{-\mathcal{W}/T} \right\rangle = e^{-\Delta F/T}. \quad (\text{A2})$$

It is to realize that, while verifying the second principle $\langle \mathcal{W} \rangle - \Delta F = \Delta S \geq 0$ thanks to the Jensen identity [21], it implies deep properties on far-from-equilibrium fluctuations. Indeed, it can be shown in many examples (*cf.* [14]) that a dominant contribution to (A2) is given by *rare* out-of-equilibrium fluctuations very far from those dominating the average $\langle \mathcal{W} \rangle$ and thus ΔS in (A1).

Appendix B: The Hatano-Sasa identity

The nonequilibrium identities have a much larger range of applicability than the one yet considered initially [13]. Among the various extensions, we will select for our purposes the one [16] which applies for purely non-equilibrium

⁸ We expect the glasma state to be similar QCD and $\mathcal{N} = 4$ SYM theories, which are both scale-invariant in the leading-log perturbative domain.

systems without reference to the notion of temperature. For this sake, one considers a situation where one has a system governed by a dynamical parameter λ , such that at each value of λ , it corresponds to a *stationary* phase space spectrum with probability distribution $\mathcal{P}^{\text{Stat}}(z; \lambda)dz$, where the variable z describes the phase-space. Then the following equality holds [16]

$$\left\langle \exp - \left[\int_{\tau_1}^{\tau_2} d\tau \frac{d\lambda}{d\tau} \frac{\partial \log \mathcal{P}(z; \lambda, \tau)}{\partial \lambda} \right] \right\rangle_{\tau_2} \equiv \int dz \exp - \left[\int_{\tau_1}^{\tau_2} d\tau \frac{d\lambda}{d\tau} \frac{\partial \log \mathcal{P}(z; \lambda, \tau)}{\partial \lambda} \right] \times \mathcal{P}(z; \lambda_2, \tau_2) \equiv 1, \quad (\text{B1})$$

where by definition $\mathcal{P}(z; \lambda, \tau) \equiv \mathcal{P}^{\text{Stat}}(z(\tau); \lambda(\tau))$ is the stationary solution for the value $\lambda(\tau)$ in the “frozen” phase space variables at time τ . Note that the resulting identity is *a priori* not dependent on the arbitrary “history” $\frac{d\lambda}{d\tau}$ of the non equilibrium mechanism.

In fact, the quantity $-\log \mathcal{P}^{\text{Stat}}(z; \lambda)$ provides a candidate definition for a quantity playing the role, after integration in phase-space of a “dynamical entropy” denoted Σ . Indeed, for the dynamical non-equilibrium process, where one starts with a state corresponding to λ_1 to a state λ_2 , the increase of ‘dynamical entropy’ gives rise to the following formula

$$\Delta \Sigma^{\lambda_1 \rightarrow \lambda_2} = \left\langle \log \frac{\mathcal{P}^{\text{Stat}}(\lambda_1)}{\mathcal{P}^{\text{Stat}}(\lambda_2)} \right\rangle_{\lambda_2} \equiv - \int \log \left\{ \frac{\mathcal{P}^{\text{Stat}}(z, \lambda_1)}{\mathcal{P}^{\text{Stat}}(z, \lambda_2)} \right\} \times \mathcal{P}^{\text{Stat}}(z; \lambda_2) dz. \quad (\text{B2})$$

In fact, our approach, see (5), Eq.(B1) is simply obtained by noting that there is no parameter dependence on time, but only *via* $\lambda \rightsquigarrow Y$ which allows for an explicit integration in (B2). Moreover, geometric scaling ensures the identity (B1) since $\mathcal{P}^{\text{Stat}}(z(\tau); \lambda(\tau))$ depends only one one variable $u(\tau) \equiv z(\lambda(\tau))$.

As can be noticed, a key feature is that no equivalent of a temperature appears in the relation (B1). In the Jarzynski case using Boltzmann factors, where temperature appears explicitly, the Hatano-Sasa identity is proven to reduce to the original Jarzynski equation (A2), namely

$$\left\langle e^{-\frac{1}{T}(W - \Delta F)} \right\rangle \equiv \left\langle e^{-\frac{1}{T}W_{\text{diss}}} \right\rangle = 1, \quad (\text{B3})$$

where W^{diss} is the dissipative part of the work.

Acknowledgments

We want to thank Kirone Mallick for useful explanations on the properties of Statistical Physics of far-from-equilibrium processes and in particular on the Jarzynski and Hatano-Sasa identities. Romuald Janik and Krzysztof Kutak are warmly thanked for works and discussions which were much stimulating for our approach.

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